Communication Systems

Lecture 3

Introduction to Signals

Signals & Systems

Signals:

A signal as the term implies, is a set of information or data. Signals are generally functions of the independent variable time.

Systems:

Signals may be processed further by systems, which may modify them or extract additional information for them.

$$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt \qquad (2.1)$$

Signal Energy

This definition can be generalized to a complex valued signal $g(t)$ as

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2\, dt \qquad (2.2)$$

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\, dt \qquad (2.3)$$

Signal Power

We can generalize this definition for a complex signal $g(t)$ as

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2\, dt \qquad (2.4)$$

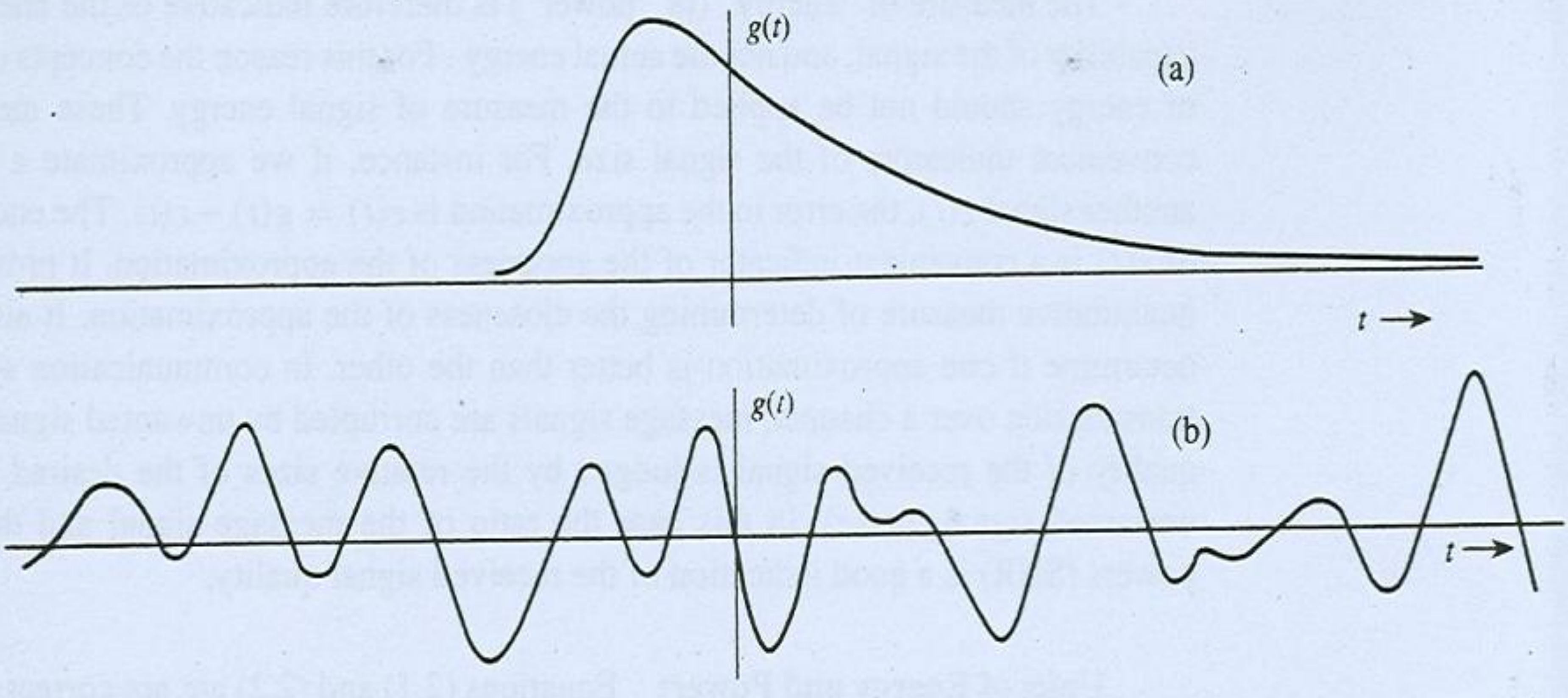
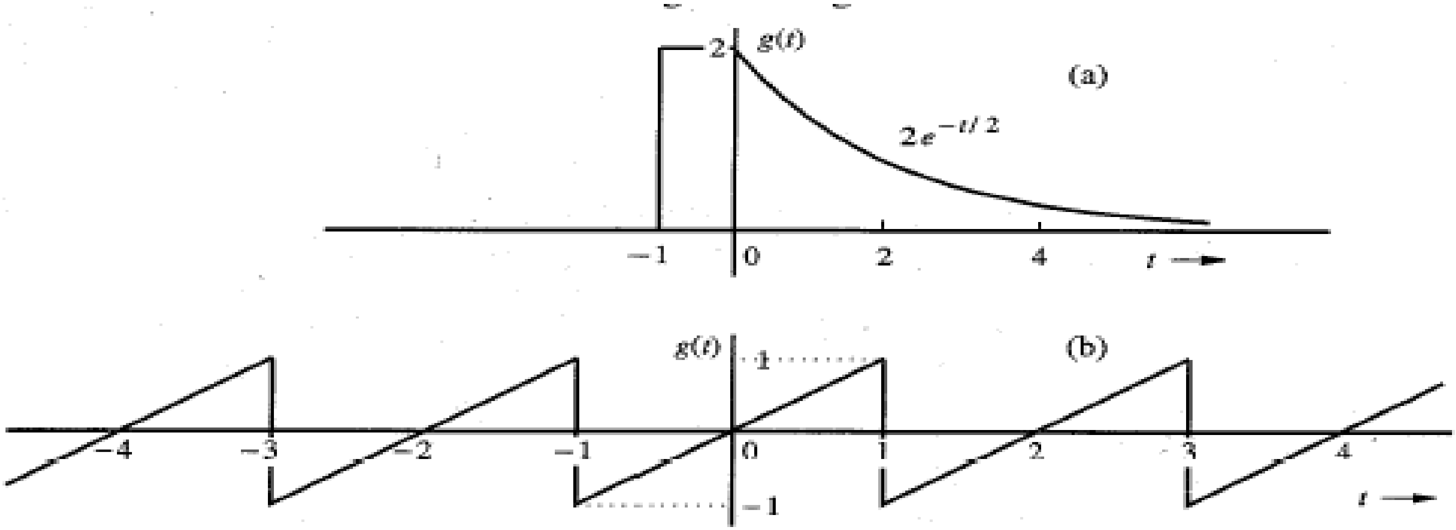


Figure 2.1 Examples of signals. (a) Signal with finite energy. (b) Signal with finite power.

Question??

- Is there any signal for which neither the energy nor the power exists?
    - YES
    - Ramp Signal

Example 2.1 Determine the suitable measures of the signals in Fig 2.3



The signal in Fig. 2.3a $\to 0$ as $|t| \to \infty$. Therefore, the suitable measure for this signal is its energy E_g given by

$$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt = \int_{-1}^{0} (2)^2\,dt + \int_{0}^{\infty} 4e^{-t}\,dt = 4 + 4 = 8$$

$$P_g = \frac{1}{2}\int_{-1}^{1} g^2(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \frac{1}{3}$$

Recall that the signal power is the square of its rms value. Therefore, the rms value of this signal is $1/\sqrt{3}$.

Classification of Signals

Continuous-Time & Discrete-Time Signals

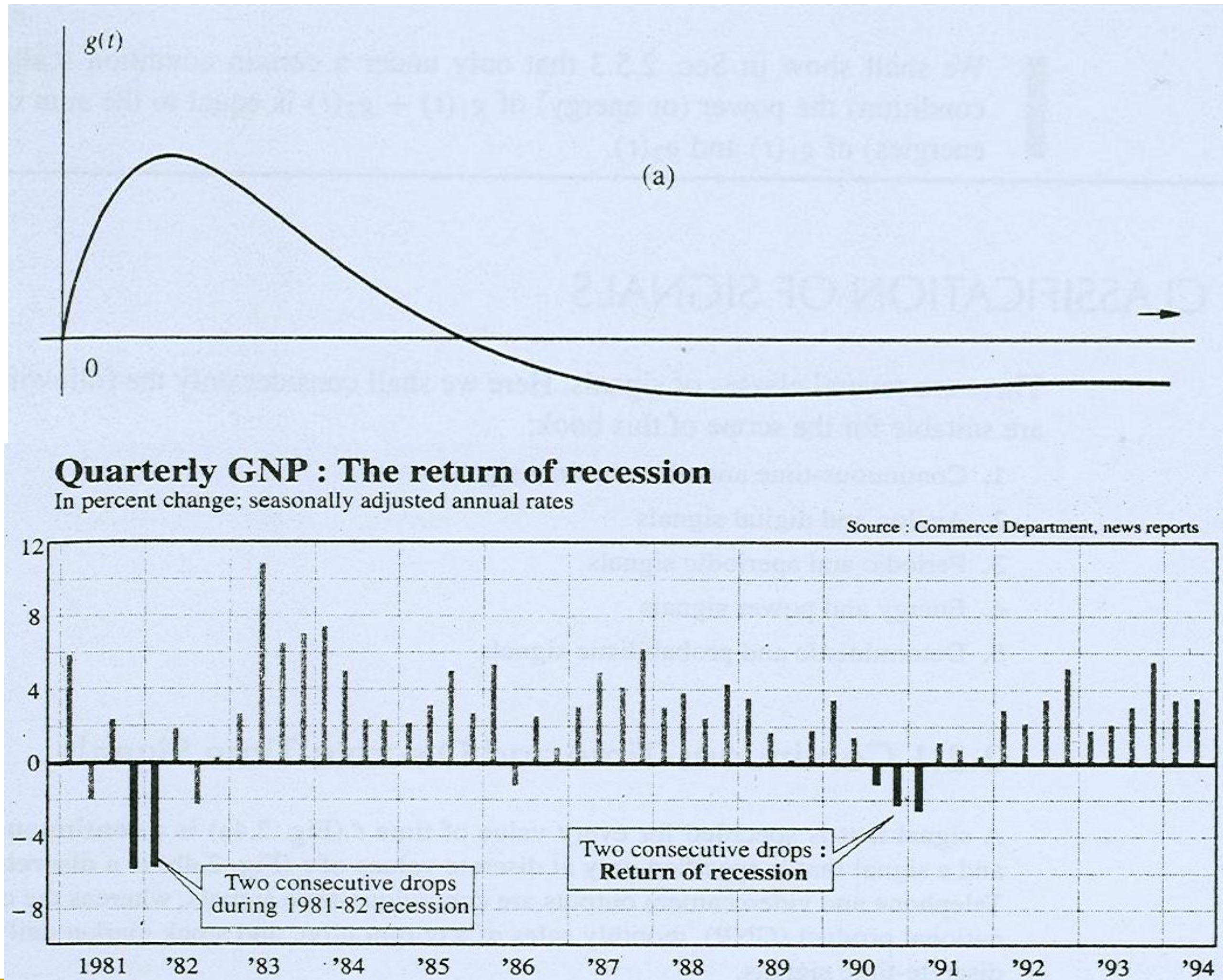


Figure 2.4 Continuous-time and discrete-time signals.

Analog & Digital Signals

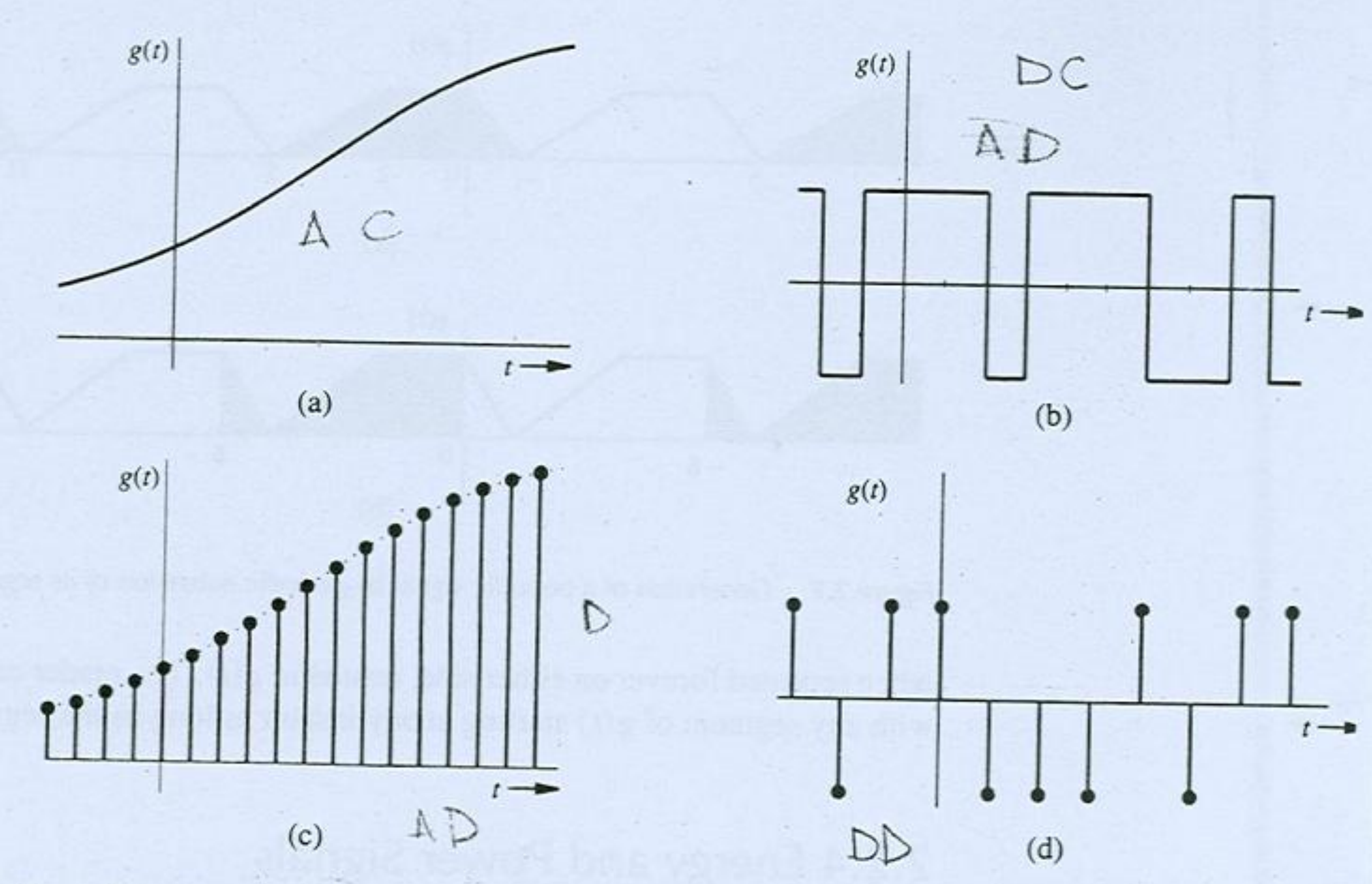


Figure 2.5 Examples of signals. (a) Analog, continuous time. (b) Digital, continuous time. (c) Analog, discrete time. (d) Digital, discrete time.

Periodic & Aperiodic Signals

A signal $g(t)$ is said to be **periodic** if for some positive constant T_0,

$$g(t) = g(t + T_0) \qquad \text{for all } t \tag{2.7}$$

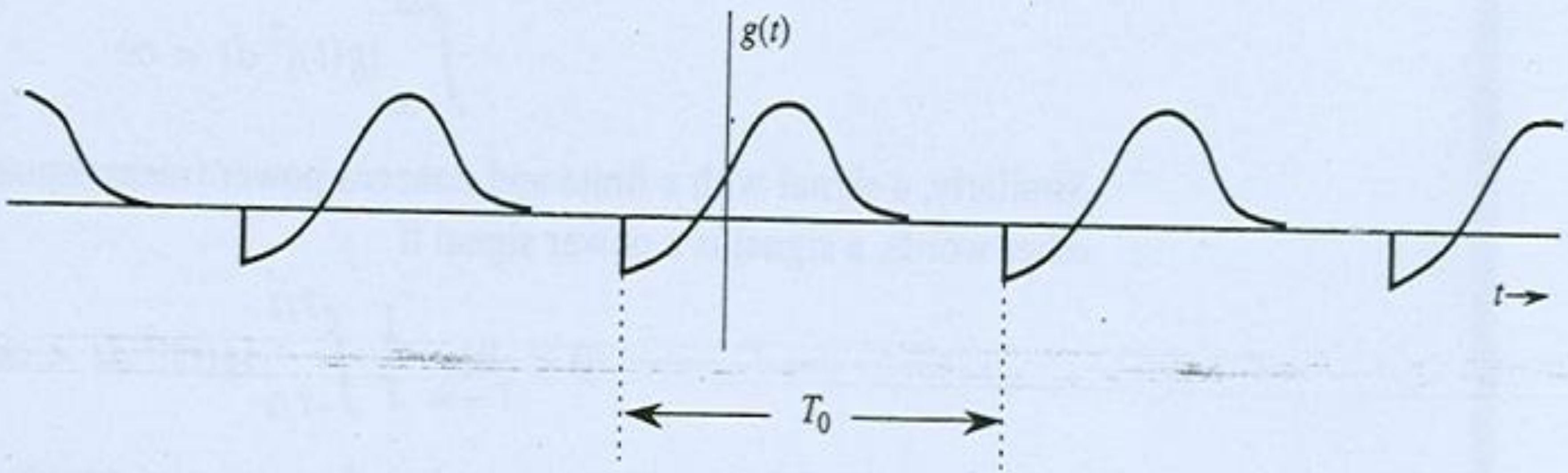


Figure 2.6 Periodic signal of period T_0.

Energy & Power Signals

A signal with finite energy is an **energy signal**, and a signal with finite power is a **power signal**. In other words, a signal $g(t)$ is an energy signal if

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty \tag{2.8}$$

Similarly, a signal with a finite and nonzero power (mean square value) is a power signal. In other words, a signal is a power signal if

$$0 < \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt < \infty \tag{2.9}$$

Deterministic & Random Signals

A signal whose physical description is known completely, in either a mathematical form or a graphical form is deterministic signal. If a signal is known only in terms of probabilistic description, such as mean value, mean squared value and so on, rather than its complete mathematical or graphical description, is a random signal. Most of the message & noise signals encountered in practice are random signals.

Some Useful Signal Operations

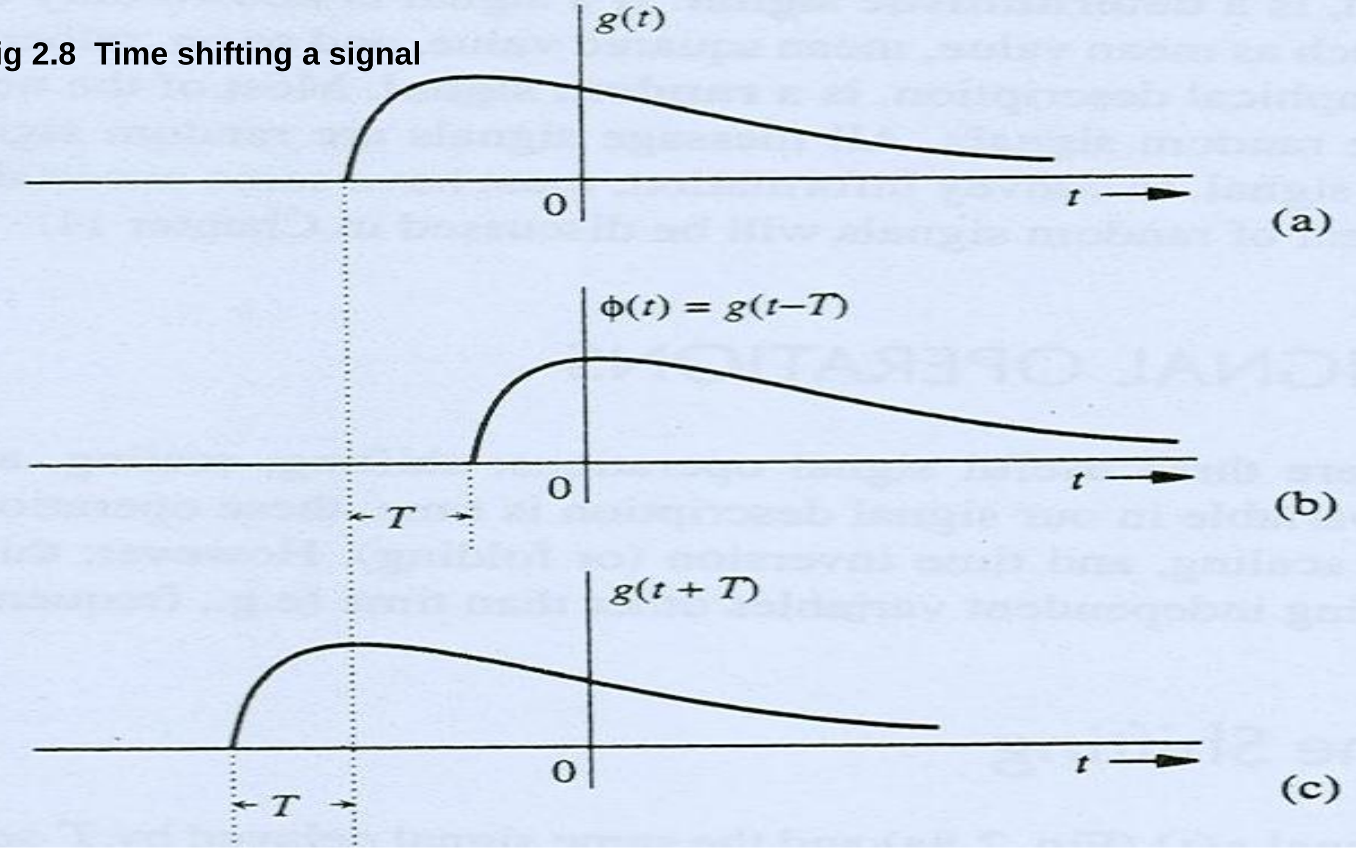
Time Shifting

$$\phi(t+T) = g(t) \tag{2.10}$$

and

$$\phi(t) = g(t-T) \tag{2.11}$$

Fig 2.8 Time shifting a signal



Time Scaling

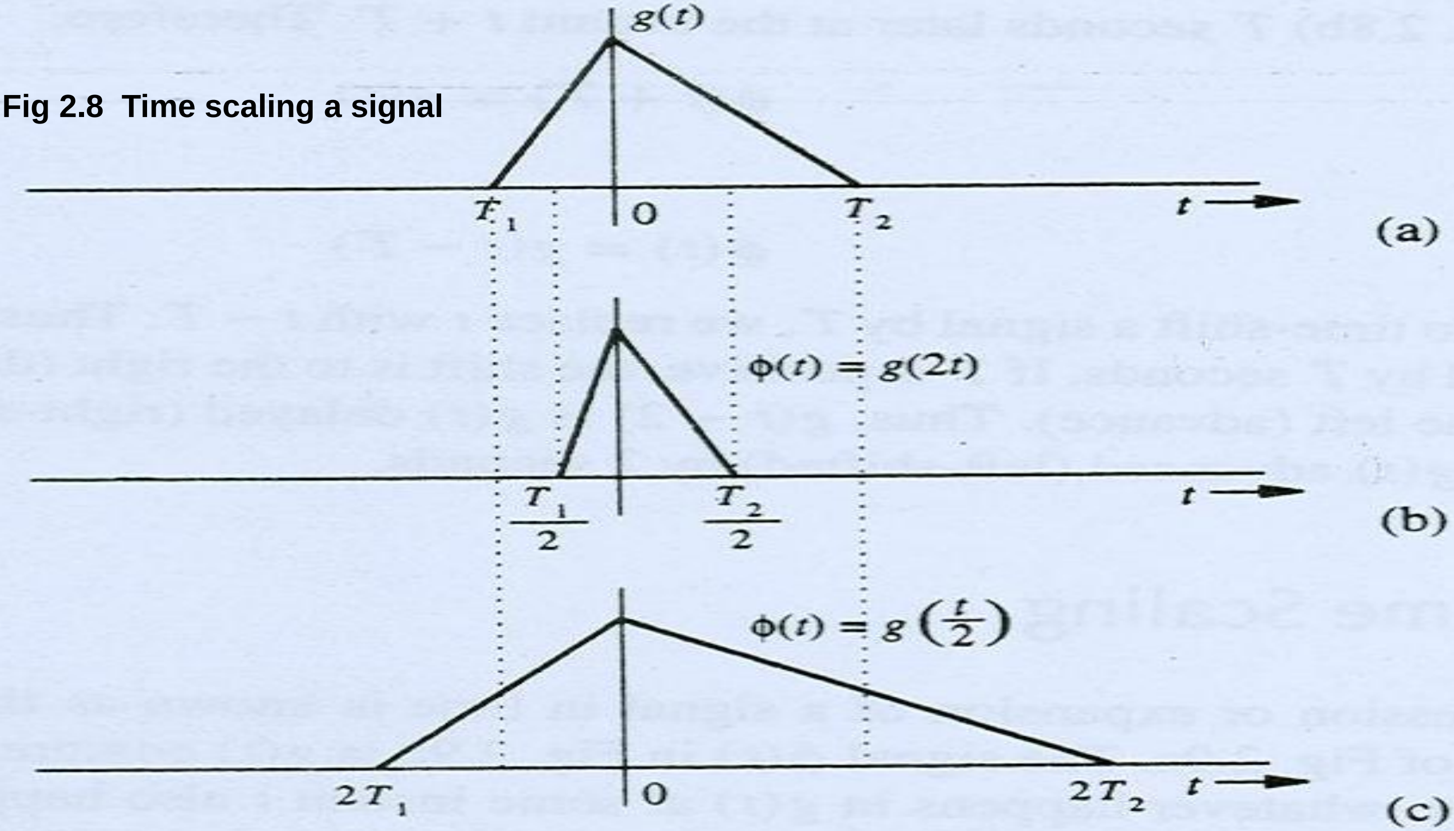
$$\phi\left(\frac{t}{2}\right) = g(t) \qquad (2.12)$$

and

$$\phi(t) = g(2t) \qquad (2.13)$$

Observe that because $g(t) = 0$ at $t = T_1$ and T_2, the same thing must happen in $\phi(t)$ at half

Fig 2.8 Time scaling a signal



Example

$$\phi(t) = g(at) \tag{2.14}$$

Using a similar argument, we can show that $g(t)$ expanded (slowed down) in time by a factor a $(a > 1)$ is given by

$$\phi(t) = g\left(\frac{t}{a}\right) \tag{2.15}$$

Example 2.3 Figure 2.10a and b shows the signals g(t) and z(t), respectively, Sketch: (a) g(3t); (b) z(t/2).

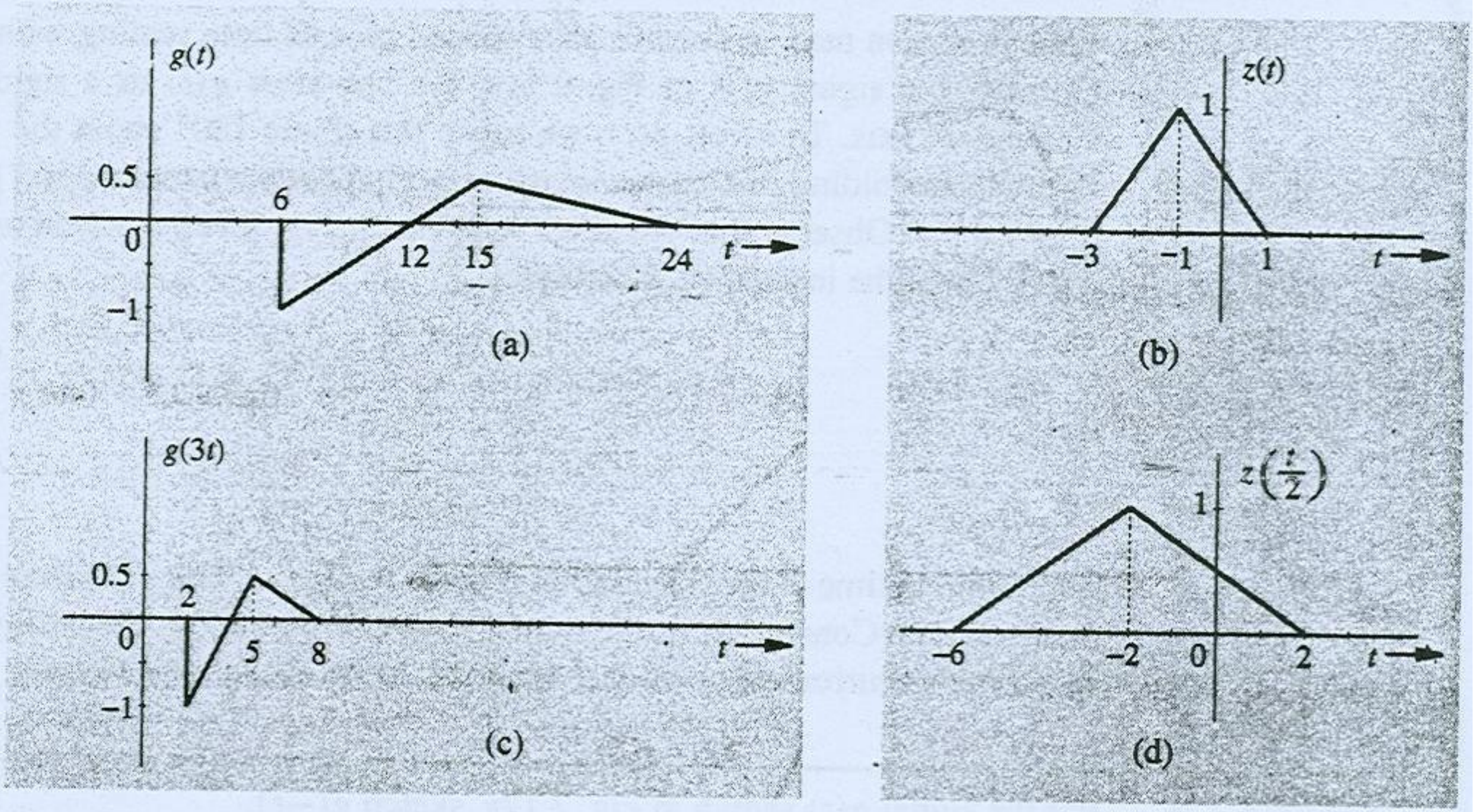


Figure 2.10 Examples of time compression and time expansion of signals.

Time Inversion (Time Reversal)

2.16

$$\phi(-t) = g(t)$$

and

$$\phi(t) = g(-t)$$

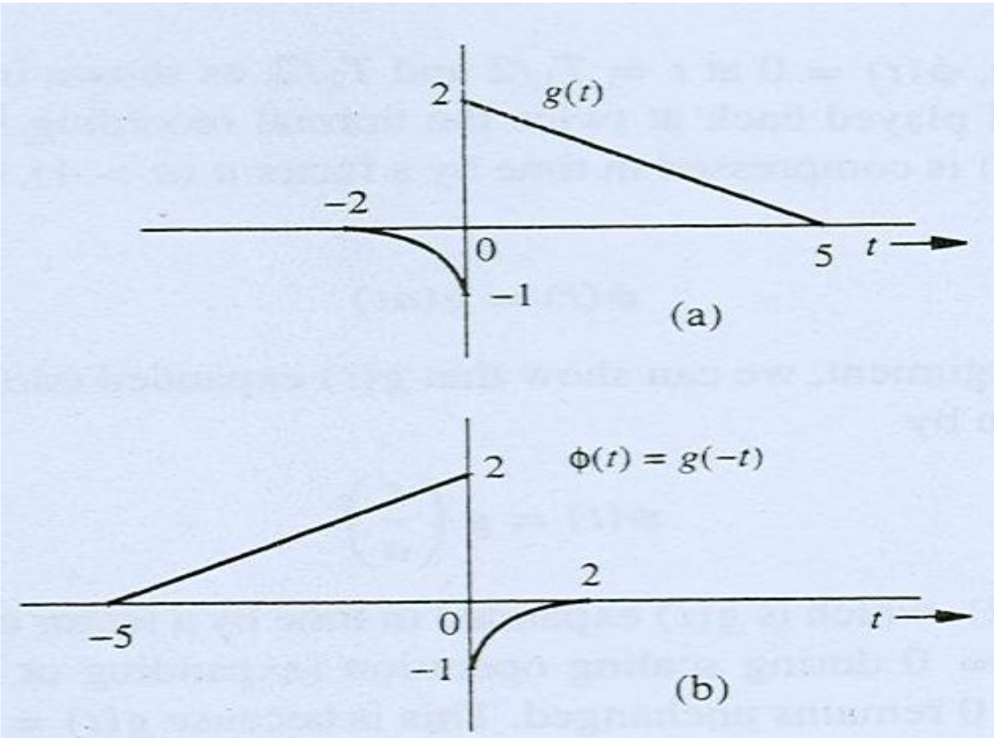


Fig 2.11 Time inversion (reflection) of a signal

Example 2.4

For the signal $g(t)$ shown in Fig. 2.12a, sketch $g(-t)$.

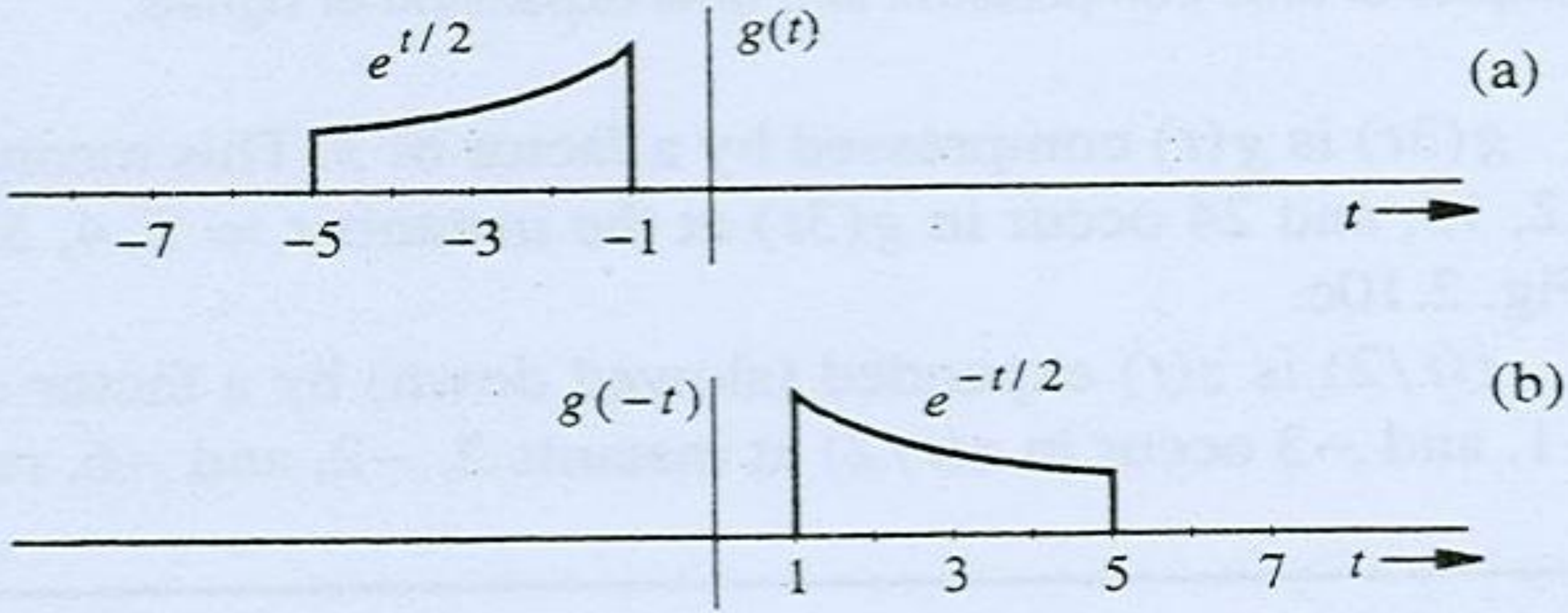


Figure 2.12 Example of time inversion.

Unit Impulse Function

The unit impulse function $\delta(t)$ is one of the most important functions in the study of signals and systems. This function was first defined by P. A. M. Dirac as

$$\delta(t) = 0 \qquad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{2.17}$$

Multiplication of a Function by an Impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t) \tag{2.18a}$$

Similarly, if $\phi(t)$ is multiplied by an impulse $\delta(t-T)$ (an impulse located at $t = T$), then

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T) \tag{2.18b}$$

provided $\phi(t)$ is continuous at $t = T$.

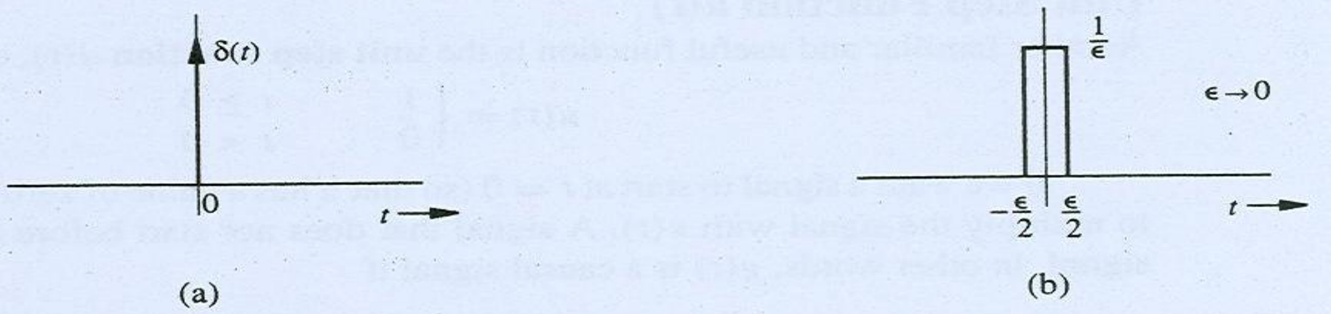


Figure 2.13 Unit impulse and its approximation.

Sampling Property of the Unit Impulse Function

From Eq. (2.18a) it follows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(0) \tag{2.19a}$$

From Eq. (2.18b) it follows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T) \tag{2.19b}$$

Unit Step Function $u(t)$

Another familiar and useful function is the **unit step function** $u(t)$, defined by (Fig. 2.14a)

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

If we want a signal to start at $t = 0$ (so that it has a value of zero for $t < 0$), we only need to multiply the signal with $u(t)$. A signal that does not start before $t = 0$ is called a **causal signal**. In other words, $g(t)$ is a causal signal if

$$g(t) = 0 \qquad t < 0$$

Signals and Vectors

- There is a perfect analogy between signals and vectors
  - Signals are not just *like* vectors. Signals *are* vectors
  - Depending on the choice of coordinate system a vector (and a signal as well) can be represented as sum of it's components in a variety of ways
- Component of a vector
  - A vector is specified by its magnitude and its direction
  - Consider two vectors **g** and **x**
  - Geometrically the components of **g** along **x** is the projection of **g** on **x**, obtained by drawing a perpendicular from the tip of **g** on the vector **x**
  - The vector **g** can be expressed in terms of vector **x** and the error vector **e** as

$$\mathbf{g}=c\mathbf{x}+\mathbf{e}$$

  - If we approximate **g** by $c\mathbf{x}$ ($\mathbf{g} \simeq c\mathbf{x}$), the error in this approximation is the vector **e**

Signals and Vectors

- Mathematically, the component of a vector **g** along vector **x** is defined as $c\mathbf{x}$, where c is chosen to minimize the length of the error vector
- The length of the component of **g** along **x**

$$c|\mathrm{x}| = |\mathrm{g}| \cos\theta$$

$$c = \frac{\mathrm{g.x}}{\mathrm{x.x}} = \frac{1}{|\mathrm{x}|^2}\mathrm{g.x}$$

- When g and x are perpendicular, then **g** has a zero component along **x**; consequently c=0, we therefore define g and x to be orthogonal if the inner product of the two vectors is zero, that is **g.x**=0

Component of a signal

- Component of a signal
  - Consider the problem of approximating a real signal $g(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$

$$g(t) \cong x(t) \qquad t_1 \le t \le t_2$$

The error $e(t)$ in this approximation is

$$e(t) = \begin{cases} g(t) - cx(t) & t_1 \le t \le t_2 \\ 0 & otherwise \end{cases}$$

For best approximation, we need to minimize the error signal, which is its energy E_e over the interval $[t_1, t_2]$ given by

$$E_e = \int_{t_1}^{t_2} e^2(t)dt = \int_{t_1}^{t_2} [g(t) - cx(t)]^2 dt$$

Component of a signal

E_e is a function of the parameter c (not t) and E_e is minimum for some choice of c

To minimize c a necessary condition is $\frac{dE_e}{dc} = 0$

$$\frac{d}{dc}\left[\int_{t1}^{t_2} [g(t) - cx(t)]^2 \, dt\right] = 0$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} g^2(t)dt\right] - \frac{d}{dc}\left[2c\int_{t_1}^{t_2} g(t)x(t)dt\right] + \frac{d}{dc}\left[c^2\int_{t_1}^{t_2} x^2(t)dt\right] = 0$$

$$-2\int_{t_1}^{t_2} g(t)x(t)dt + 2c\int_{t_1}^{t_2} x^2(t)dt = 0$$

$$c = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt} = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)dt$$

Component of a signal

E_e is a function of the parameter c (not t) and E_e is minimum for some choice of c

To minimize c a necessary condition is $\dfrac{dE_e}{dc} = 0$

$$\frac{d}{dc}\left[\int_{t1}^{t_2}[g(t)-cx(t)]^2\,dt\right]=0$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} g^2(t)dt\right]-\frac{d}{dc}\left[2c\int_{t_1}^{t_2} g(t)x(t)dt\right]+\frac{d}{dc}\left[c^2\int_{t_1}^{t_2} x^2(t)dt\right]=0$$

$$-2\int_{t_1}^{t_2} g(t)x(t)dt+2c\int_{t_1}^{t_2} x^2(t)dt=0$$

$$c=\frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt}=\frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)dt$$

Component of a signal

The optimum value of c that minimizes the energy of error signal in this approximation

- Now comparing the two expressions

$$c = \frac{\mathrm{g.x}}{\mathrm{x.x}} = \frac{1}{|\mathrm{x}|^2}\mathrm{g.x} \qquad c = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt} = \boxed{\frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)dt}$$

- Area under the product of two signals corresponds to the inner product of two vectors

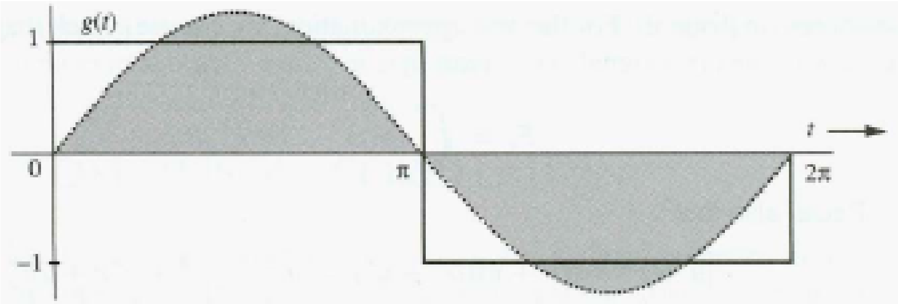
- The energy of a signal is the inner product of a signal with itself and corresponds to the vector length squared (which is the inner product of the vector with itself)

Component of a signal

- If the component of a signal $g(t)$ along another signal $x(t)$ is zero (that is $c=0$), then signals $g(t)$ and $x(t)$ are orthogonal over the interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} g(t)x(t)dt = 0$$

  - Example
    - Approximate $g(t)$ in terms of $\sin t$



Orthogonality in Complex Signals

- In complex signal, the energy of error signal, E_e is given as:

$$E_e = \int_{t_1}^{t_2} |g(t) - cx(t)|^2 dt$$

$$|u+v|^2 = (u+v)(u^* + v^*) = |u|^2 + |v|^2 + u^*v + uv^*$$

- Using this result, after some manipulation, we get

$$E_e = \int_{t_1}^{t_2} |g(t)|^2 dt - \left|\frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)x^*(t)dt\right| + \left|c\sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)x^*(t)dt\right|^2$$

- The first two terms on the right hand side are independent of c, E_e is minimized by choosing c such that the third term is zero, which yields

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x^*(t)dt$$

Orthogonality in Complex Signals

- Two complex functions $x_1(t)$ and $x_2(t)$ are orthogonal over an interval $t_1 \le t \le t_2$ if

$$\int_{t_1}^{t_2} x_1(t) x_2^*(t) dt = 0 \qquad or \qquad \int_{t_1}^{t_2} x_1^*(t) x_2(t) dt = 0$$

  - This is a general definition of the orthogonality, of which the real signals are a special case

Orthogonality in Complex Signals

- Energy of the sum of orthogonal signals
  - The length (squared) of the sum of two orthogonal vectors is equal to the sum of lengths squared of the two vectors
    - If vectors x and y are orthogonal and if z=x + y, then $|z|^2 = |x|^2 + |y|^2$
  - The energy of the sum of two orthogonal signals is equal to the sum of the energies of the two signals
    - If signals $x(t)$ and $y(t)$ are orthogonal over an interval $[t_1, t_2]$ and if $z(t) = x(t) + y(t)$, then $E_z = E_x + E_y$

  We can now prove this for complex signals as:

$$|u+v|^2 = (u+v)(u^* + v^*) = |u|^2 + |v|^2 + u^* v + u v^*$$

$$\int_{t_1}^{t_2} |x(t) + y(t)|^2 dt = \int_{t_1}^{t_2} |x(t)|^2 + \int_{t_1}^{t_2} |y(t)|^2 + \int_{t_1}^{t_2} x(t) y^*(t) dt \int_{t_1}^{t_2} x(t)^* y(t) dt$$

$$= \int_{t_1}^{t_2} |x(t)|^2 + \int_{t_1}^{t_2} |y(t)|^2$$